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Estimation and Control of Error Based on P-Convergence

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i

TABLE OF CONTENTS

ESTIMATION AND CONTROL OF ERROR BASED ON P-CONVERGENCE

SUMMARY

The relative error in energy can be estimated on the basis of the asymptotic estimate of the rate of p-convergence. Nearly optimal control of error can be achieved when feedback information, generated from p-convergence data, is used for deciding whether the mesh should be refined or the polynomial degree of elements increased.

1.   INTRODUCTION

In order to keep the discussion in focus, we shall be concerned only with problems of plane elasticity and finite element models based on the displacement formulation. In a number of cases the domain has reentrant corners or sudden changes occur in the boundary conditions. In the neighborhood of such points the exact solution is of the form:

$$\underline{u} = \sum_{i=1}^{\infty} A_i \, r^{\alpha_i} \, \underline{F}_i(\theta), \qquad \alpha_i > 0 \tag{1}$$

where $\underline{u}$ is the displacement vector, r and θ are polar coordinates centered on the point, α_i are determined from the condition that the solution must satisfy the Navier-Lame equations and the boundary conditions on the edges that meet at the corner [1]. For the purposes of the following discussion we shall assume that α_i have been ordered such that $\alpha_1 \leq \alpha_2 \leq \alpha_3$ etc. $\underline{F}_i$ are sinusoidal (vector) functions. A_i are unknown coefficients (amplitudes), closely related to the stress intensity factors in linear elastic fracture mechanics. In fact, in linear elastic fracture mechanics $\alpha_1 = \frac{1}{2}$, $\alpha_2 = \frac{1}{2}$ and $A_1 = K_I/\sqrt{2\pi}$, $A_2 = K_{II}/\sqrt{2\pi}$, K_I, K_{II} being, respectively, the Mode I, Mode II stress intensity factors. We shall be interested in the lowest value of α_i, specifically α_1. Some typical α_1 values that frequently occur in engineering practice are shown in Fig. 1. When α_1 is large, or when α_1, α_2,... are integers, or when the points of singularity lie outside of and far from the solution domain then the solution is said to be smooth.
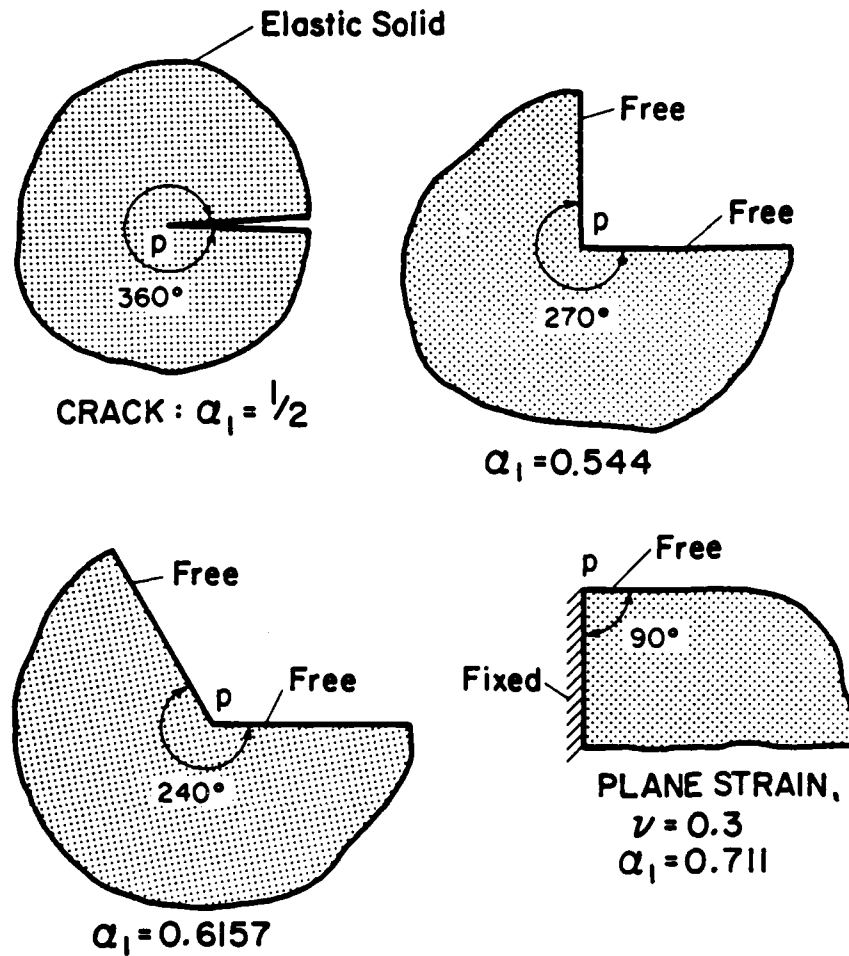
Fig. 1

Some α_1 values for isotropic elastic solids.

The finite element solution $\underline{u}_{FE}$ minimizes the potential energy expression with respect to a set of functions that can be written in the following form:

$$\underline{u} = \sum_{i=1}^{N} \phi_i(x,y) \, \underline{\delta}_i \tag{2}$$

where $\phi_i(x,y)$ are the basis functions, constructed from the element shape functions in such a way that the appropriate continuity and boundary

conditions are satisfied. The basis functions depend on the choice of the finite element mesh for the solution domain, the shape functions defined on the finite elements and the mapping of the elements:

$$x = x(\xi, \eta), \qquad y = y(\xi, \eta) \tag{3}$$

where ξ and η are the standard coordinates and (3) represents transformation of the standard element into the 'real' element. The element shape functions are defined on the standard element. We shall be concerned only with polynomial shape functions. Unless the transformation is isoparametric or subparametric however, the mapped shape functions and therefore the basis functions are not polynomials. The set of all functions that can be written in the form of eq. (2) is called a finite element space.

A particular choice of mesh and polynomial degree p may not yield an approximate solution of sufficient precision. It is necessary therefore to have the ability to improve the quality of approximation. This involves increasing the number of degrees of freedom. When the number of degrees of freedom is increased in such a way that the finite element spaces of fewer degrees of freedom are embedded in the finite element spaces of greater number of degrees of freedom then the process of increasing the degrees of freedom is called *extension*. The algorithm or strategy by which the number of degrees of freedom is increased is called *extension operator*.

Extension operators are classified into three major categories called *h-extension*, *p-extension*, and *h-p extension*. When aspects of implementation are emphasized then the word *version* rather than *extension* is used. In the h-version the polynomial degree of the elements is fixed, therefore extension is possible only by mesh refinement. The size of the elements is usually denoted by h, hence the name: h-version. In the p-version the polynomial degree of the elements may be varied over a substantial range. Therefore extension is possible through increasing the polynomial degree of elements. In the h-p extension the number of degrees of freedom is increased by some combination of mesh refinement and increase of polynomial degree.

2. CONVERGENCE IN ENERGY NORM

Finite element solutions minimize the error in energy norm. Therefore the magnitude of the error in energy norm is a good measure of the overall quality of approximation. Asymptotic error estimates are available for the three methods of extension. Following is a brief summary:

In the h-extension when a sequence of meshes is obtained through uniform refinement then the estimate of error is:

$$||e||_E \leq \frac{k_1}{N^{1/2 \min(p,\alpha)}} \qquad (4)$$

where $||e||_E$ is the error in energy norm (i.e. the square root of the strain energy of the error); N is the number of degrees of freedom; p is the polynomial degree of elements and α measures the smoothness of the exact solution. Specifically, in the problems considered herein, α is the smallest exponent of r in eq. (1), i.e. $\alpha = \alpha_1$. The exponent of N is called the asymptotic rate of convergence.

When the sequence of meshes is obtained in such a way that the error associated with each element is approximately the same then the meshes are called equilibrated meshes and the estimate is:

$$||e||_E \leq \frac{k_2}{N^{p/2}} \qquad (5)$$

Note that the rate of convergence is independent of α and can be made arbitrarily large by choosing sufficiently large p.

In p-extension the mesh is fixed and the polynomial degree of elements is increased. In this case, when the point of singularity is located at a nodal point, the estimate is:

$$||e||_E \leq \frac{k_3}{N^{\alpha}} \qquad (6)$$

Note that the rate of convergence is exactly twice the rate of convergence of the h-extension based on uniform mesh refinement.

In the h-p extension optimal mesh refinement is coupled with optimal p-distribution and the estimate is

$$||e||_E \leq \frac{k_4}{\exp(\gamma N^\theta)} \tag{7}$$

where γ, θ are constants, $\theta \geq 1/3$. In this case the asymptotic rate of convergence is exponential.

Some of these results are illustrated graphically in Fig. 2. The relative errors shown in Fig. 2 are typical values.
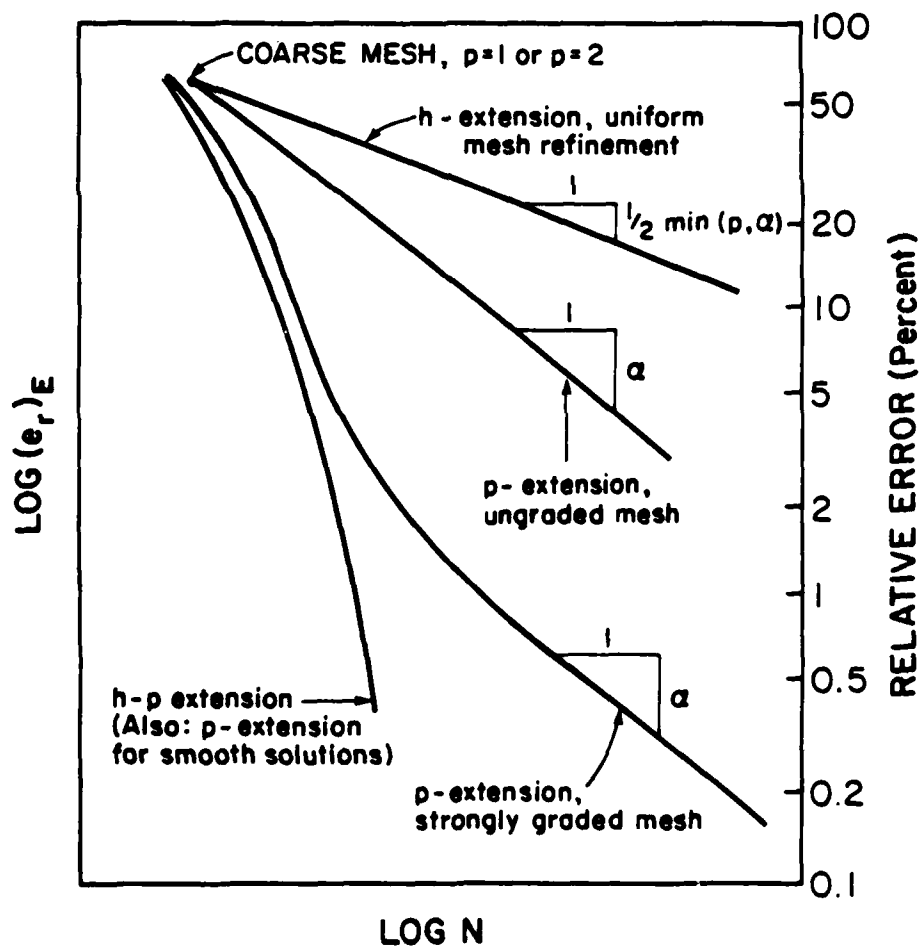

Fig. 2

Performance of the h-, p- and h-p extension processes.
(The relative error values shown are typical for certain engineering problems).

The asymptotic rate of convergence of p-extensions was discussed in a number of papers and several examples were presented [2-7]. In [3] a rigorous proof of eq. (6) was given. In the examples the meshes were usually designed in such a way that the minimum number of elements needed for representing the domain were used. An important exception was a demonstration of the h-p extension in [4].

In this paper it will be shown that: (1) the exponential rate of convergence is in fact realized in the case of problems with smooth solutions when the p-extension is used; (2) Problems with stress singularities can be made to behave almost as problems with smooth solutions do when properly graded meshes are used in conjunction with the p-extension and (3) the relative error in energy norm can be estimated on the basis of the asymptotic estimate for the p-extension, eq. (6).

3. PROBLEMS WITH SMOOTH SOLUTIONS

3.1 Thick walled cylinder under internal pressure.

Let us first consider a thick walled cylinder under internal pressure. In this case the classical solution is known. (See, for example, [8]). Assuming plane strain conditions and Poisson's ratio of 0.3, the strain energy U for a 45 degree sector, as shown in Fig. 3, is $0.748746249 \; p^2 r_i^2/E$ per unit length of the cylinder, given of course that $r_o/r_i = 2$. E is the modulus of elasticity.

The relative error in energy norm is:

$$(e_r)_E = \sqrt{\frac{U-U_p}{U}} \tag{8}$$

where: U exact strain energy

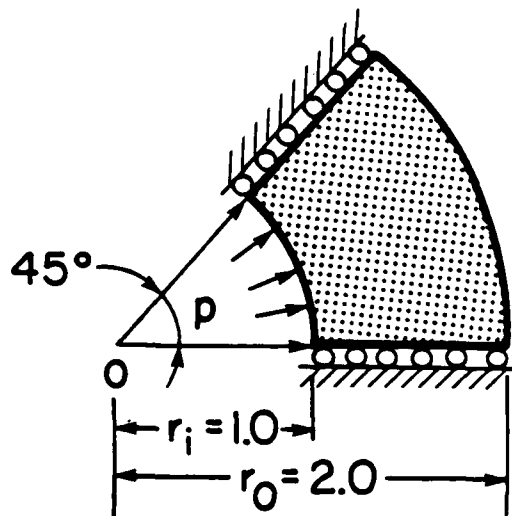U_p computed strain energy, polynomial degree p.

Fig. 3

45 degree sector of a thick walled
cylinder under internal pressure.

Using a single quadrilateral element, mapped by linear blending so that the annular sector is represented exactly, $(e_r)_E$ was computed for $p = 1,2,...8$. The results are shown in Figs. 4 and 5. In Fig. 4 the relative error is plotted against the number of degrees of freedom, N, on log-log scale. It is seen that the slope of the curve increases with N. This is consistent with the qualitative illustration of convergence in Fig. 2. In Fig. 5 the relative error is plotted against the number of degrees of freedom on semi-log scale. In this case the slope of the relative error approaches a constant value, indicating that the rate of convergence is nearly exponential.

Note that $p = 4$ is more than sufficient for engineering accuracy for this problem.
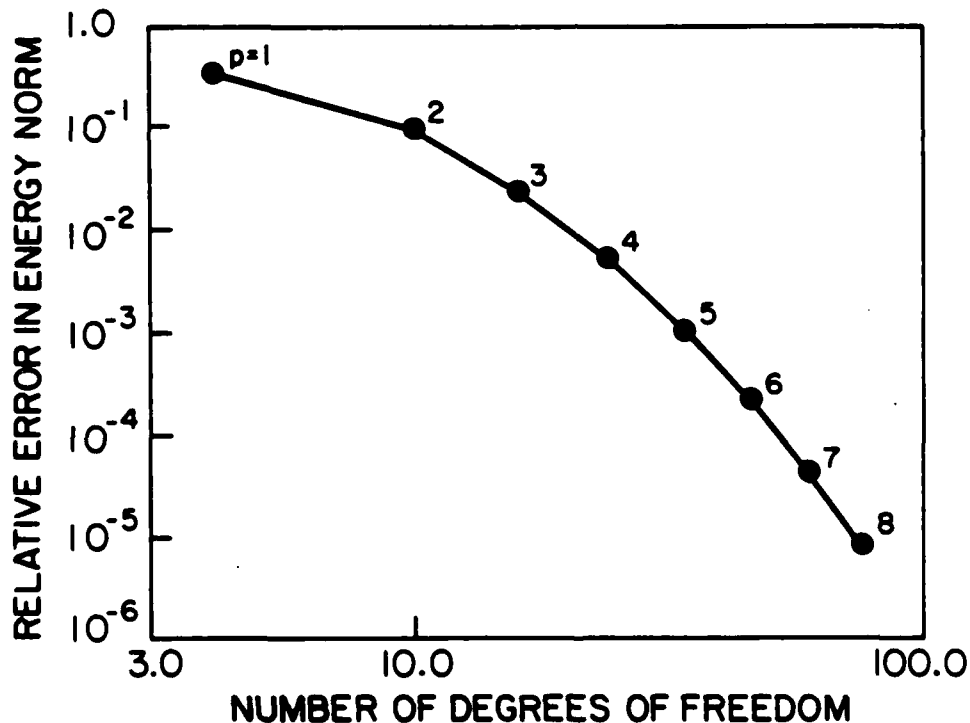

Fig. 4

45 degree sector of a thick walled cylinder under internal pressure. Relative error in energy norm plotted against N on log-log scale.
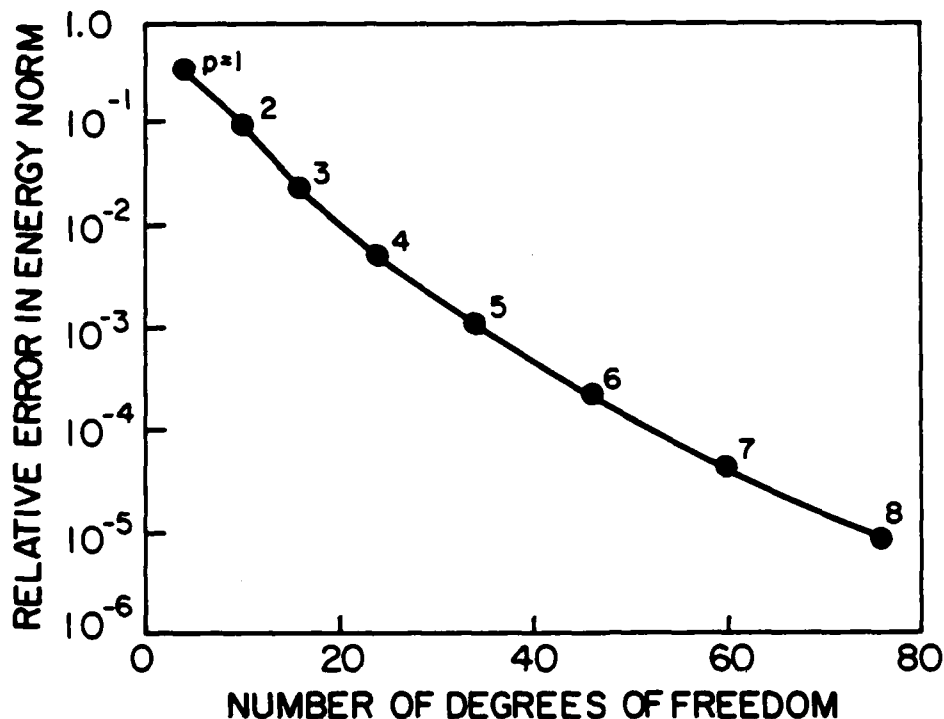
Fig. 5

45 degree sector of a thick walled cylinder under internal pressure.
Relative error in energy norm plotted against N on semi-log scale.

3.2 Circular hole in a rectangular panel

Let us next consider the well known problem of a circular hole in a
rectangular panel subjected to uniaxial tension. See, for example [9].
Because the solution of this problem is smooth (no reentrant corners are
present and no sudden changes in material properties or loading conditions
occur) the minimum number of elements needed for representing the geometry
is the best choice for the p-version. In the h-version the error is con-
trolled by mesh refinement. A typical h-version mesh (produced by a
practicing engineer) and the p-version mesh are shown in Fig. 6. Of
interest is the maximum stress. The maximum stress, computed directly
from the finite element solution using the three element mesh with the
polynomial degree ranging from 1 to 8 and $\frac{r}{w}$ = 0.5, is shown in Fig. 7.

7. ACKNOWLEDGEMENTS

6. SUMMARY AND CONCLUSIONS

When properly graded meshes are used then the performance of the
p-extension in the pre-asymptotic range is very close to the best per-
formance attainable by sequences of optimal meshes coupled with sequences
of optimal p-distributions.

The p-extension process provides means for estimating the relative
error in energy norm. It also provides information on the basis of
which it is possible to decide whether the mesh should be refined or the
polynomial degree of elements should be increased.

When the exact solution is smooth then the coarsest possible mesh
should be used. In this case mesh design is controlled entirely by the
geometry of the domain. The only restriction on the mesh is that the
mapping of the elements must be smooth also, therefore large aspect
ratios must be avoided. In the p-extensions aspect ratios as large as 20:1
are permissible, however.

When the exact solution is not smooth, for example corner singular-
ities are present, then the points of singularity must be isolated by
one or more layers of small elements. Grading of the elements should be
such that the element sizes are in geometric progression with the smallest
element(s) at the point of singularity. The geometric progression
should have a common factor of about 0.15. In this case entry into the
asymptotic range is shifted toward higher p-values and the pre-asymptotic
convergence of the p-extension process is much stronger than asymptotic.
When the mesh is properly designed then the finite element solution
behaves almost as if the exact solution were smooth.

In general, overrefinement in the neighborhood of singular points
is preferable to underrefinement.

Proper mesh design is highly beneficial from the point of view of
stress computations: The oscillatory behavior of stresses can be attenuated
to such an extent that, with the exception of the immediate neighborhood of
singular points, the error is very small everywhere.

TABLE IV

Estimated and true error in energy norm.
Cracked panel, Mesh B.

P	N_p	$\dfrac{U_p E}{K_I^2 a}$	β	Percent Rel. Error Est.'d	Percent Rel. Error True
1	27	0.207859	–	–	35.10
2	78	0.233048	–	–	13.02
3	137	0.236076	1.71	8.89	6.46
4	220	0.236739	2.55	3.46	3.71
5	327	0.236895	2.87	1.76	2.68
6	458	0.236953	2.22	1.48	2.17
7	613	0.236983	1.66	1.42	1.86
8	792	0.237001	1.26	1.43	1.64

Note that the preasymptotic range is extended when Mesh B is used.

Systematic application of the procedure just described is the 'feedback h-p extension' in the terminology of reference [12].

It is seen that β is monotonically increasing. Although the relative errors in energy norm are very small, the estimated and true errors are close.

5.2 Cracked panel

We first tabulate the convergence data for Mesh A (see Table III). The observed rate of convergence (β) is clearly stronger than the asymptotic rate $(2\alpha_1 = 1)$ up to $p = 6$. For higher p values the rate of convergence becomes weaker and approaches the asymptotic rate. Note also that the estimated and true error values are reasonably close.

At $p = 8$ the relative error in energy norm is 4.22 percent relative error in strain energy. From the practical point of view this level of precision is usually more than sufficient. For example, the relative error in the stress intensity factor K_I, computed from the strain energy release rate at $p = 8$, is only 0.38 percent. Nevertheless, if greater

TABLE III

Estimated and true error in energy norm.
Cracked panel, Mesh A.

p	N_p	$\dfrac{U_p E}{K_I^2 a}$	β	Percent Rel. Error Est.'d	Percent Rel. Error True
1	18	0.201706	–	–	38.62
2	52	0.229199	–	–	18.21
3	94	0.234117	1.31	13.38	11.15
4	152	0.235489	1.96	6.11	8.15
5	226	0.236053	1.57	5.26	6.53
6	316	0.236346	1.33	4.70	5.51
7	422	0.236525	1.10	4.49	4.77
8	544	0.236643	1.02	4.11	4.22

precision is needed, then the mesh should be refined using the geometric progression rule. Repeating the procedure for Mesh B shown in Fig. 11, the results shown in Table IV were obtained.

the h-extension process. In fact, the relative error for the cantilever problem shown in Fig. 10 was estimated in this way. When p-extension is used then $\beta = 2\alpha_1$. When h-extension is used with uniform mesh refinements then $\beta = \min(\alpha_1, p)$.

Referring to Fig. 2, we note that when the mesh is well designed then the extension is not in the asymptotic range. Therefore the meshes should be designed in such a way that the asymptotic range is not entered when the polynomial degree is increased. Ideally, the mesh should be designed so as to cause β to increase with N. When three successive finite element solutions are available in the extension process, i.e. U_p, U_{p-1}, U_{p-2} and N_p, N_{p-1}, N_{p-2} are known, then we can estimate β by assuming C to be constant and solving the following equation for β:

$$\frac{U_p - U_{p-1}}{N_{p-1}^{-\beta} - N_p^{-\beta}} = \frac{U_{p-1} - U_{p-2}}{N_{p-2}^{-\beta} - N_{p-1}^{-\beta}} \tag{14}$$

If we find that β is not increasing with p and the error is still large then those elements that have vertices on the singular point should be refined, using the geometric progression rule for refinement. The procedure is illustrated on the basis of two problems: one has a very smooth solution, the other is characterized by a strong stress singularity.

5.1 Thick walled cylinder under internal pressure.

The problem was solved with a single finite element which is shown in Fig. 3. The convergence data are listed in Table II.

TABLE II

Estimated and true error in energy norm.
Thick walled cylinder under internal pressure.

p	N_p	$\dfrac{U_p E}{p^2 r_1^2}$	β	Percent Rel. Error Est.'d	True
1	4	0.6539035818	–	–	35.5905
2	10	0.7421867856	–	–	9.3598
3	16	0.7483608276	2.6312	5.8098	2.2688
4	24	0.7487274398	5.9418	0.6954	0.5012
5	34	0.7487454062	7.3680	0.1413	0.1061
6	46	0.7487462140	8.8350	0.0283	0.0217
7	60	0.7487462477	10.4389	0.0055	0.0043
8	76	0.7487462491	12.0693	0.0010	0.0008

5. ESTIMATES OF ERROR IN ENERGY NORM

The p-extension process can be used for obtaining valuable (feedback) information concerning the overall quality of approximation. On the basis of this information it is possible to estimate the relative error in energy norm and decide whether it is better to refine the mesh or increase the polynomial degree of elements.

The asymptotic estimates of error in energy norm were reviewed in Section 2. Equivalently, the error in strain energy can be written as:

$$\left| U-U_p \right| \leq CN_p^{-\beta} \tag{10}$$

where U is the exact strain energy, U_p is the computed strain energy, N_p is the corresponding number of degrees of freedom, subscript p represents the polynomial degree of elements, C is a positive constant and β is the rate of convergence. In the case of homogeneous kinematic boundary conditions $U > U_p$ and we can write:

$$U-U_p \leq CN_p^{-\beta}. \tag{11}$$

When the finite element solution is in the asymptotic range and β is known then we can readily compute C from:

$$C = \frac{U_p - U_{p-1}}{N_{p-1}^{-\beta} - N_p^{-\beta}} \tag{12}$$

and therefore U can be estimated from:

$$U = \frac{U_p N_p^\beta - U_{p-1} N_{p-1}^\beta}{N_p^\beta - N_{p-1}^\beta}. \tag{13}$$

Although the exact solution is not known, the strain energy of the exact solution can be estimated with great precision through either the p- or

Since the exact solution is known, we are in position to examine the accuracy of stress computations. We have computed the stresses at $x = 0.25a$ $y = 0.25a$ (See Fig. 11). Because this point lies at the boundary of two elements, we obtain two sets of stress values, depending on which element is used for the stress computations. We refer the element with vertices at $(0,0)$, (a,a) $(0,a)$ as element 1, the element with vertices at $(0,0)$ $(a,0)$, (a,a) as element 2. The computed σ_y values for both elements, together with the corresponding relative errors, are shown in Table I. Note that the error is very small for a wide range of p-values.

TABLE I

Stress σ_y computed at $x = 0.25a$, $y = 0.25a$,
using Mesh A. The exact value of σ_y is: $0.8390\ K_I/\sqrt{a}$

	Element 1		Element 2	
p	$\sigma_y\sqrt{a}/K_I$	Rel. Err. (%)	$\sigma_y\sqrt{a}/K_I$	Rel. Err. (%)
1	0.7891	5.95	0.5804	30.8
2	0.7336	12.6	0.7516	10.4
3	0.8355	0.42	0.8510	1.43
4	0.8456	0.79	0.8411	0.25
5	0.8402	0.14	0.8407	0.20
6	0.8359	0.37	0.8383	0.08
7	0.8352	0.45	0.8362	0.33
8	0.8359	0.37	0.8361	0.35

Alternative means of stress computation are available [10,11]. The results shown here indicate however that direct computation of stresses from finite element solutions is adequate for practical purposes when the mesh is properly designed. This is especially important in the case of design computations where the overall behavior of the solution is of interest. The extraction techniques presented in [10,11] and investigated in [6,7] are of great practical importance when the solution in the immediate neighborhood of a stress singularity is of interest.

$$\sigma_y = \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left[1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2}\right] , \quad -\pi \le \theta \le \pi \qquad (9b)$$

$$\tau_{xy} = \frac{K_I}{\sqrt{2\pi r}} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \frac{3\theta}{2} , \quad -\pi \le \theta \le \pi. \qquad (9c)$$

Therefore the exact solution is known and the strain energy can be computed. It is $0.23706469 \ K_I^2 a/E$ for the half panel. (Due to symmetry, only half of the panel has to be analyzed).

The p-convergence paths for Meshes A and B are shown in Fig. 12. Note that the convergence path corresponding to Mesh B is very similar to the qualitative behavior shown in Fig. 2 for strongly graded meshes.
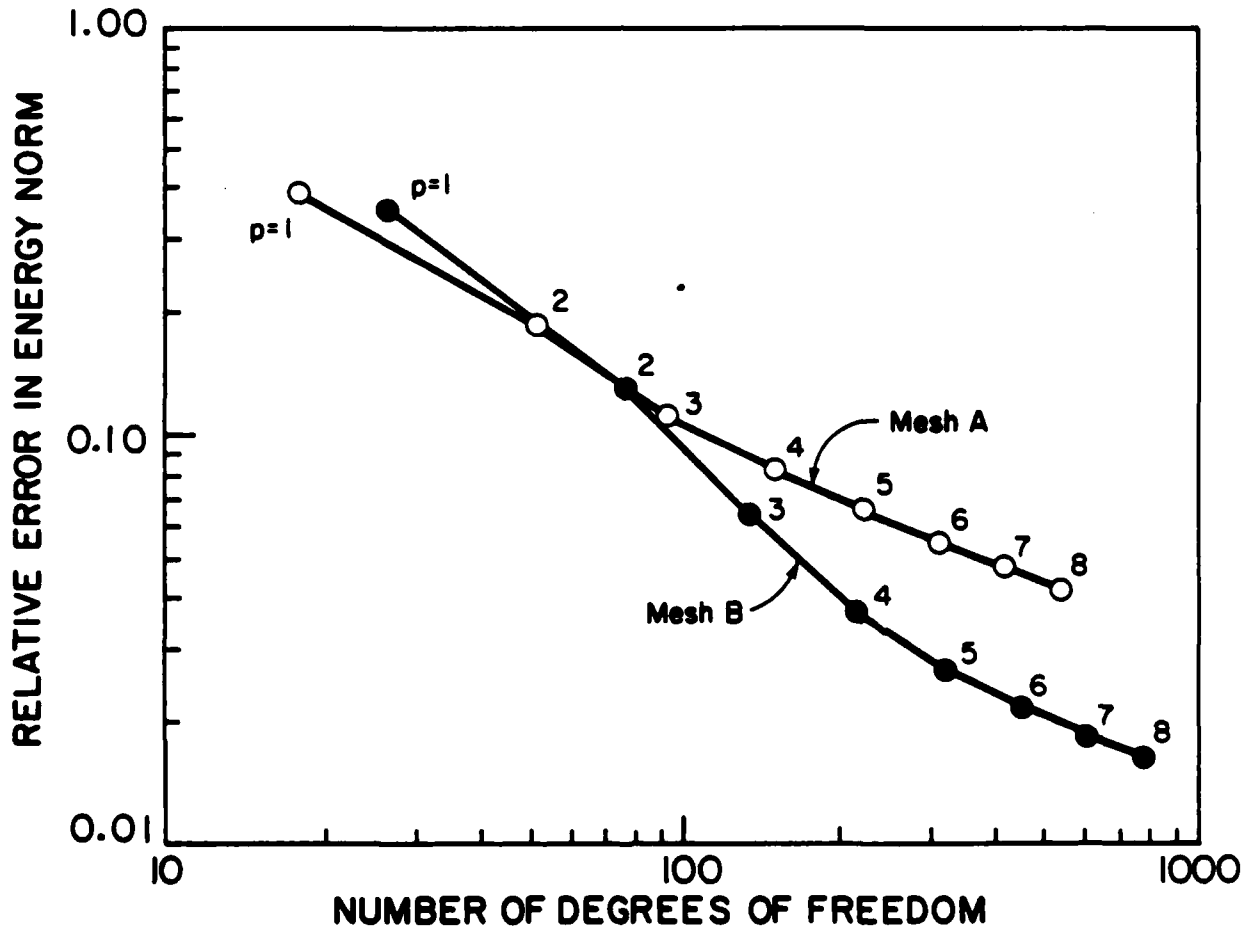

Fig. 12

Cracked panel. Performance of the p-extension
when strongly graded meshes are used.

in the number of degrees of freedom when two elements are added is not substantial (Fig. 10). Similarly, the use of graded p-distribution, i.e. assigning low polynomial degrees to the small elements at the singularities and high polynomial degrees to the large elements away from the singularities, as in the h-p extension, would not produce appreciable savings in practical computations.

4.2 Cracked Panel

Let us now consider the cracked panel shown in Fig. 11.
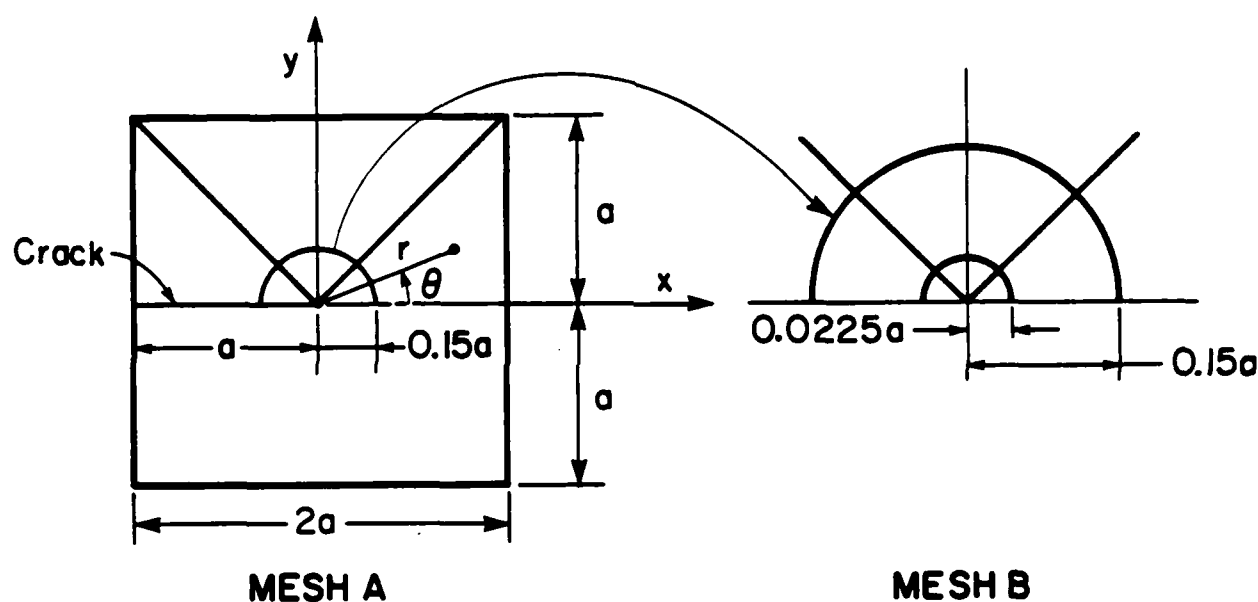

Fig. 11

Cracked panel.

We assume plane strain conditions, Poisson's ratio of 0.3, and unit thickness and load the panel in such a way that the tractions exactly correspond to the well known first term of the asymptotic expansion for Mode I, i.e.:

$$\sigma_x = \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left[1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2}\right], \quad -\pi \leq \theta \leq \pi \qquad (9a)$$

The rates of convergence in energy norm are shown in Fig. 10 for the h-extension with uniform mesh refinement (p=1), and the p-extension for three different mesh layouts. It is seen that the theoretically predicted rate is realized in the case of both the h- and p-versions when the mesh is uniform, i.e. Mesh A is used. In either case it is impractical to achieve 1 percent relative error in energy norm. The rate of convergence is so small that the h- and p-extension processes do not provide effective control of error in this norm. On the other hand, the convergence of the p-extension process is very strong when strongly graded meshes are used. Mesh B5 is a five-element mesh, shown in Fig. 9, Mesh B7 is a seven-element mesh, the grading of the elements is in geometric progression toward the singularities, the common factor being 0.1. The strong convergence obtained with these meshes seemingly contradicts the theoretically predicted rate of eq. (6). This can be explained as follows: The estimates given by eqs. (4) to (7) are asymptotic estimates, hence they are valid only for 'large' N values. The size of N for the estimates to hold depends on the mesh design. When the elements at singular points are large then the error of approximation is controlled by the error associated with these elements and N does not have to be very large for the asymptotic estimates to govern. When the elements at the singular points are small, as in the case of Mesh B5, the error associated with the larger elements away from the singularity, where the solution is smooth, is the controlling factor. Consequently the finite element solution behaves as if the exact solution were smooth. Entry into the asymptotic range occurs only when the polynomial degree is sufficiently high so that the elements at singular points begin to control the error of approximation. The validity of this explanation is confirmed by the results obtained with Mesh B7: Refinement at the singularity does not significantly reduce the error. In fact, for low p-values it is more efficient to use Mesh B5 than Mesh B7. The cross-over occurs at about p = 5.

The strong preasymptotic behavior of the p-extension associated with Meshes B5, B7 is similar to the asymptotic behavior of problems with smooth solutions.

Note that overrefinement (Mesh B7) is not detrimental from the point of view of roundoff error and is not very wasteful. The increase
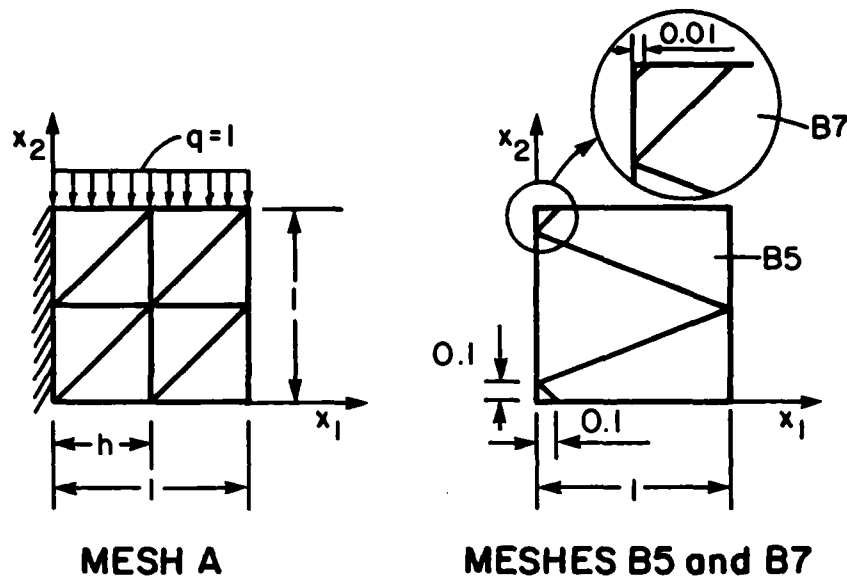
MESH A   **MESHES B5 and B7**
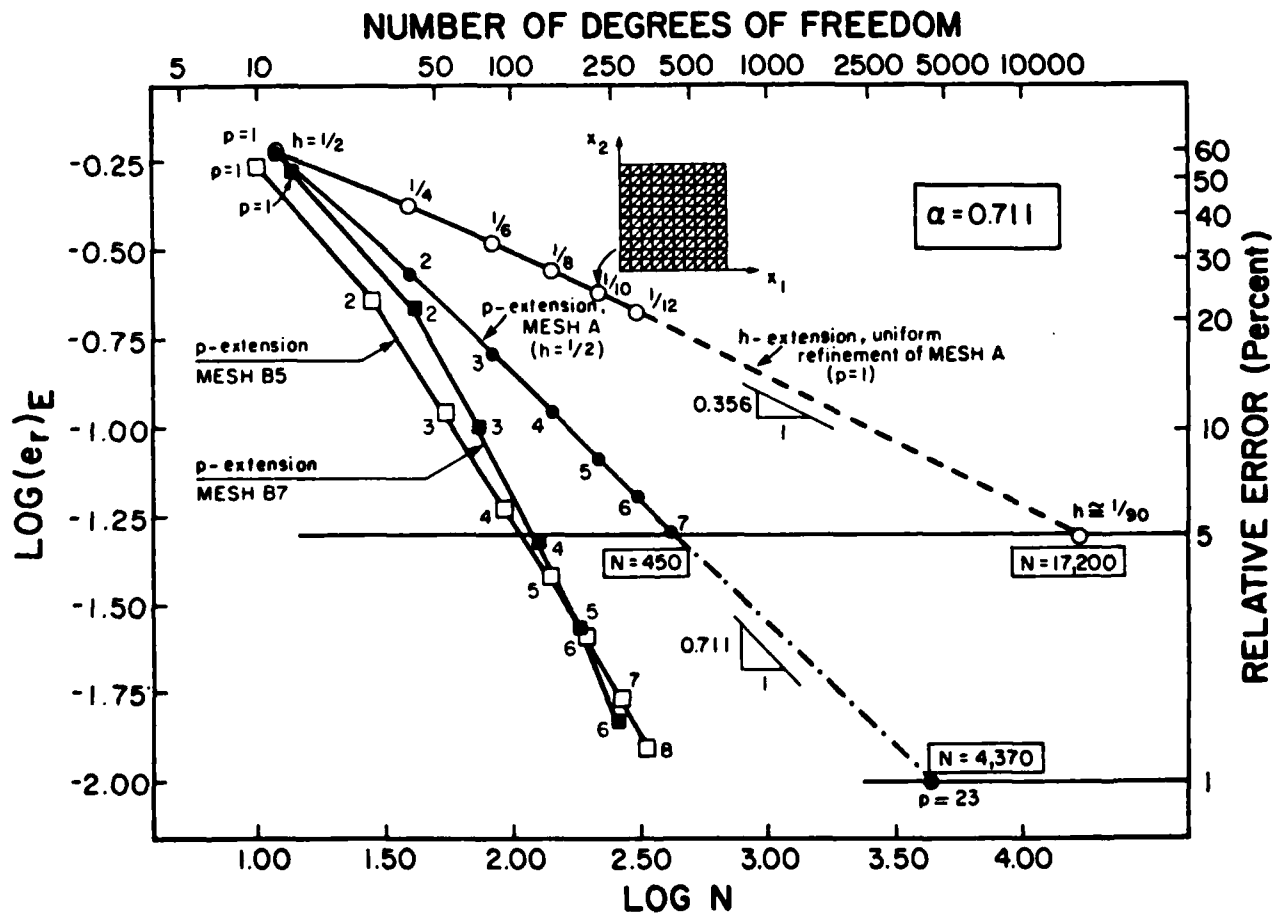
Fig. 9

Short cantilever beam.

Fig. 10

Short cantilever beam.
Performance of the h- and p-extensions.

4. PROBLEMS WITH STRESS SINGULARITIES

When the minimum number of elements is used and stress singularities are present (i.e. the form of the exact solution at one or more points is that of eq. (1) with α_1 less than one) then entry into the asymptotic range occurs at low p-values, usually $p = 3$ or $p = 4$, and the stress field exhibits strong oscillatory behavior. This oscillatory behavior is closely related to the smoothness of the exact solution: The smoother the exact solution, the less pronounced are the oscillations. These oscillations are beneficial in the sense that the faster rate of convergence of the p-extension in energy (as compared with the h-extension based on uniform meshes) is owed to the property of polynomials that they are able to oscillate with increased frequency as the polynomial degree is increased and the wavelength of oscillations decreases with distance measured from element boundaries. On the other hand, stress oscillations are confusing when stresses are of primary interest. Numerical experiments have shown, however, that the boundaries of finite elements attenuate stress oscillations. The attenuation is so substantial that with proper mesh design the error in stress maxima, outside of the immediate neighborhood of stress singularities, can be reduced to under 1 percent for all stress analysis problems of practical importance.

In the h-p extension the meshes are strongly graded toward the singularity; the element sizes are reduced in geometric progression with a common factor of about 0.15 and the polynomial degrees of elements are assigned such that the smallest element has the lowest polynomial degree, the largest element the highest. Nearly as good performance can be achieved when the elements are strongly graded toward the singularity but the distribution of polynomial degrees is uniform. In the following we demonstrate the performance of the p-extension on the basis of two examples.

4.1 Short Cantilever Beam

The geometric definition of the problem and the meshes used in our analysis are shown in Fig. 9. Plane strain conditions and Poisson's ratio of 0.3 are assumed.

It is seen that there are minor oscillations in the computed stresses but for p = 4 to 8 the stress value is well within the 5 percent relative error range. For p = 8 the finite element solution is identical with the solution presented in [9]. More importantly, the extension process confirms the fact that convergence has occurred. The variation in stresses is 0.3 percent in the range of p = 6,7,8. Given the fact that the solution is smooth and therefore convergence (in energy) is strong, one can reasonably estimate that the relative error is under 1 percent.

The fact that the p-version tolerates large aspect ratios is illustrated in Fig. 8. Using the same three element mesh as shown in Fig. 6, the stress concentration factor was computed for a wide range of r/w ratios. It is seen that substantial deviation from the theoretical value occurs only at very large and very small r/w ratios. The aspect ratios of the finite elements are very large in those cases, however.
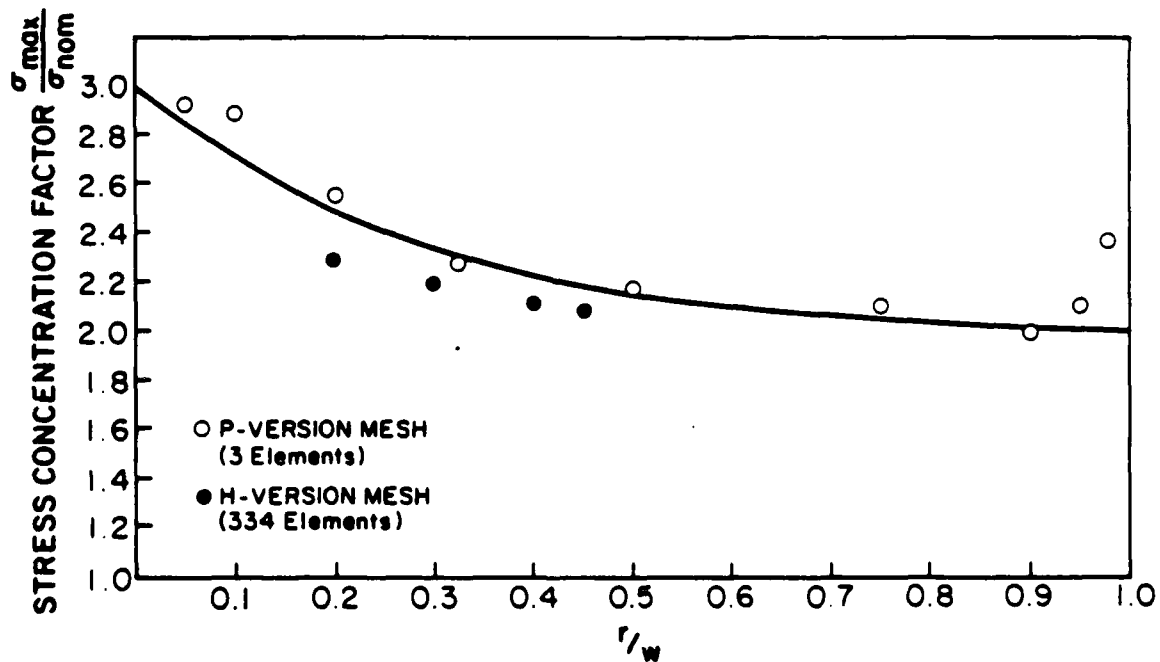

Fig. 8

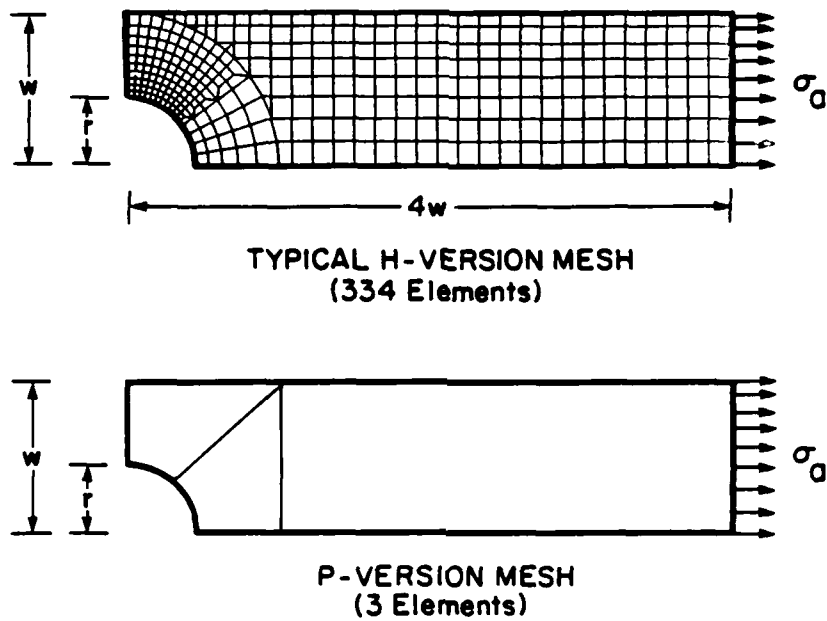Stress concentration factor as function of $\frac{r}{w}$.

TYPICAL H-VERSION MESH
(334 Elements)

P-VERSION MESH
(3 Elements)

Fig. 6

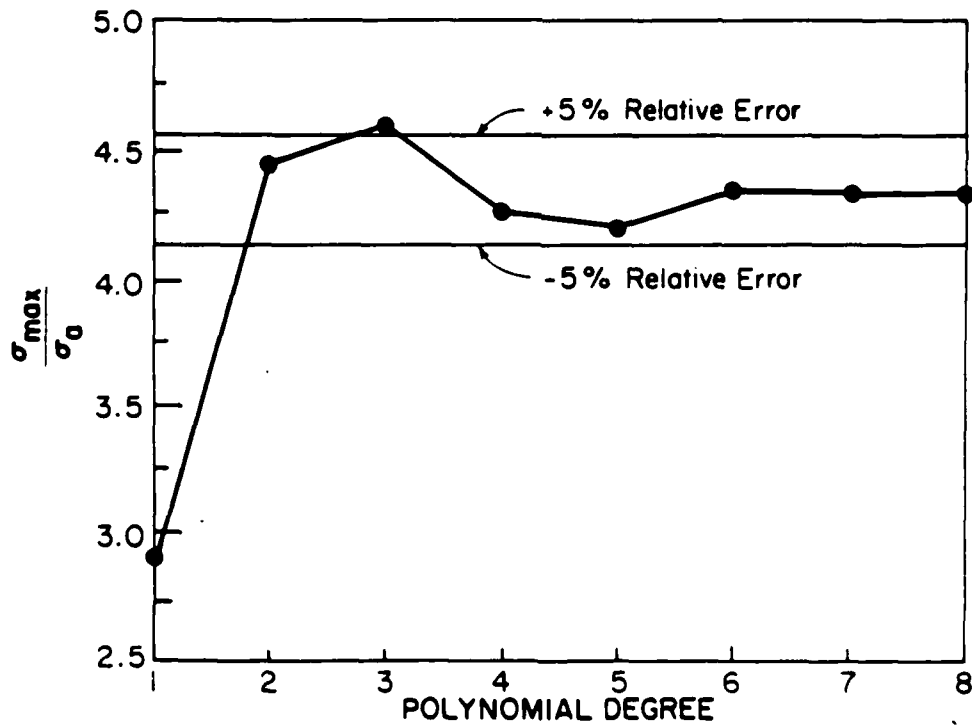Quarter model of a rectangular panel with a circular hole.

Fig. 7

Ratio of maximum stress to applied stress, $\frac{r}{w} = 0.5$.

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